Chapter Five
Second Order Linear Equations

1. Introduction

In this chapter we are going to consider single equations of order two and higher. Since almost all differential equations that occur in applications are of order one or two, most of the emphasis will be on equations of order two. Moreover, we will consider only linear equations.

The most general linear ordinary differential equation of order two has the form,

$$a(t)y''(t) + b(t)y'(t) + c(t)y(t) = f(t). \qquad (1.1)$$

We call this a linear equation because the unknown function $y(t)$ and its derivatives appear in the equation in a linear way. That is, there are no products of y and its derivatives, no powers higher than one, no nonlinear functions with y or its derivatives as an argument. But there is a more satisfactory way to define what we mean by a linear equation. First, define a "function" into which we substitute functions of t rather than numbers,

$$L[\bullet] = a(t)\frac{d^2}{dt^2}[\bullet] + b(t)\frac{d}{dt}[\bullet] + c(t)[\bullet]. \qquad (1.2)$$

i.e.,

$$L[y_1(t)] = a(t)\frac{d^2}{dt^2}[y_1(t)] + b(t)\frac{d}{dt}[y_1(t)] + c(t)[y_1(t)],$$
$$L[y_2(t)] = a(t)\frac{d^2}{dt^2}[y_2(t)] + b(t)\frac{d}{dt}[y_2(t)] + c(t)[y_2(t)].$$

We refer to $L[\bullet]$ as an operator. Then it is easy to see that

$$\begin{aligned} L[y_1(t)+y_2(t)] &= a(t)\frac{d^2}{dt^2}[y_1+y_2] + b(t)\frac{d}{dt}[y_1+y_2] + c(t)[y_1+y_2] \\ &= a(t)\frac{d^2}{dt^2}[y_1] + a(t)\frac{d^2}{dt^2}[y_2] + b(t)\frac{d}{dt}[y_1] + b(t)\frac{d}{dt}[y_2] + c(t)[y_1] + c(t)[y_2] \\ &= L[y_1(t)] + L[y_2(t)]. \end{aligned}$$

Similarly, for any constant k,

$$\begin{aligned} L[ky(t)] &= a(t)\frac{d^2}{dt^2}[ky(t)] + b(t)\frac{d}{dt}[ky(t)] + c(t)[ky(t)] \\ &= ka(t)\frac{d^2}{dt^2}[y(t)] + kb(t)\frac{d}{dt}[y(t)] + kc(t)[y(t)] \\ &= kL[y(t)] \end{aligned}$$

These two observations can be combined in the following single assertion,

$$L[C_1y_1(t) + C_2y_2(t)] = C_1L[y_1(t)] + C_2L[y_2(t)], \qquad (1.3)$$

which holds for all constants C_1, C_2 and all functions $y_1(t), y_2(t)$. Any operator L having property (1.3) is said to be a **linear operator**. A differential equation, like (1.1), for which the associated operator, (1.2), is linear is said to be a linear differential equation. The word linear derives from the fact that a real valued function $f(x)$ with the property that $f(C_1x_1 + C_2x_2) = C_1f(x_1) + C_2f(x_2)$, has a straight line for its graph of f(x) versus x. The new definition of linear differential equation is equivalent to the previous definition.

Note that the linearity condition (1.3) implies that if $y_1(t), y_2(t)$ are both solutions of the homogeneous equation $L[y(t)] = 0,$ then the combination $C_1 y_1(t) + C_2 y_2(t)$ is also a solution of the same equation for all choices of the constants $C_1, C_2.$ This is known as the **principle of superposition** and it holds for all linear homogeneous equations.

What we have just observed is that the set of all solutions to $L[y(t)] = 0$ is a subspace. Recall that we encountered subspaces back in the chapter on linear algebra where we defined a subspace of R^n as a set of vectors that is closed under the operation of forming linear combinations. The principle of superposition is just the assertion that the set of all solutions to $L[y(t)] = 0$ is closed under the operation of forming linear combinations and is therefore a subspace.

Here we are not dealing with vectors but rather a set of functions, $y(t),$ defined for $t \geq 0,$ which satisfy the homogeneous differential equation $L[y(t)] = 0.$ This includes the tacit assumption that the functions all have first and second order derivatives so that they can be substituted into the second order operator, $L[\bullet].$ We will define this set, $S,$ as the **solution space** for the differential operator, L. That is, $S = \{y(t) : L[y] = 0\}$ so S is analogous to the null space of a matrix $A.$ We will be interested in the dimension of the solution space $S.$ We recall that the dimension of a subspace is equal to the maximum number of linearly independent vectors in the subspace and we previously defined linear independence for vectors. The meaning of linear independence for functions is essentially the same as for vectors in R^n.

Definition *A set of functions* $\{y_1(t), y_2(t), \ldots, y_N(t)\},$ *all of which are defined on an interval,* $I,$ *are said to be linearly independent if the following statements are equivalent:*

1. $C_1 y_1(t) + C_2 y_2(t) + \ldots, + C_N y_N(t) = 0 \quad \textit{for all } t \textit{ in } I$
2. $C_1 = C_2 = \ldots = C_N = 0.$

It is obvious that 2. always implies 1. but the converse is only true if the functions are linearly independent on the interval, I. Essentially, a set of functions is linearly independent if none of the functions can be expressed as linear combinations of the remaining functions. Now we need a method for determining if a collection of vectors is or is not linearly independent. For this purpose we define

$$W[y_1, y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t) y_2'(t) - y_2(t) y_1'(t).$$

Here $W[y_1, y_2](t)$ is called the "Wronskian determinant" for the functions y_1 and y_2. Then we can prove the following results:

Lemma 1.1

If $y_1, y_2 \in S$ then either $W[y_1, y_2](t) = 0$ for all t, or else $W[y_1, y_2](t)$ is never zero for any t.

Lemma 1.2

Functions $y_1, y_2 \in S$ are independent if and only if $W[y_1, y_2](t) \neq 0.$

The proofs of the results in this section will be found in the appendix to this chapter. For now, it is only necessary to state the results.The next lemma is an auxiliary result whose only purpose is to prove the lemmas about existence and uniqueness of solutions to $L[y(t)] = 0.$ Note that lemma 1.3 implies that if $y \in S$ satisfies $y(0) = y'(0) = 0,$ then $y(t)$ is zero for all $t \geq 0$.

Lemma 1.3
If $y \in S$ then

$$Ce^{-kt} \leq \sqrt{y(t)^2 + y'(t)^2} \leq Ce^{kt}$$

where

$$C = \sqrt{y(0)^2 + y'(0)^2} \qquad \text{and} \qquad k = 1 + |b| + |c|.$$

The next lemma asserts that there cannot be more than one solution to the initial value problem for L.

Lemma 1.4
For arbitrary constants, A, B there exists **at most** one $y \in S$ such that $y(0) = A \text{ and } y'(0) = B$.

Combining lemmas 1.4 and 1.5 allows us to assert that

For arbitrary constants, A, B there exists one and only one solution for

$$L[y(t)] = 0 \quad \textit{with} \quad y(0) = A \textit{ and } y'(0) = B.$$

Lemma 1.5
For arbitrary constants, A, B there exists **at least** one $y \in S$ such that $y(0) = A \text{ and } y'(0) = B$.

Finally, we have a lemma which tells us the dimension of S, it is 2.

Lemma 1.6
Let $y_1, y_2 \in S$ be linearly independent. Then every $y \in S$ can be written uniquely in the form $y(t) = C_1 y_1(t) + C_2 y_2(t)$.

The assertion of this lemma is that there are at least two independent functions in S but any set of three or more functions in S must be dependent. That is, S is a two dimensional subspace of the smooth functions defined for $t \geq 0$.

Now that we have this information about the set of solutions to a second order linear ODE, we begin the process of finding the solutions.

2. Equations With Constant Coefficients- Homogeneous Solution

An equation of the form (1.1) is still more difficult to solve than we are prepared to handle in this course. We will consider the simpler situation in which the coefficients in the equation are all constants. This is an equation of the form,

$$ay''(t) + by'(t) + cy(t) = f(t). \qquad (2.1)$$

and we will begin by considering the even simpler situation where the equation is homogeneous,

$$ay''(t) + by'(t) + cy(t) = 0. \qquad (2.2)$$

To solve this equation we note that if $y(t) = e^{rt}$ is substituted into (1.5) we get

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = (ar^2 + br + c)\,e^{rt} = 0,$$

and this is satisfied if and only if r solves $ar^2 + br + c = 0.$ If the roots of this quadratic equation are denoted by r_1 and r_2 then e^{r_1t} and e^{r_2t} are both solutions of the equation (2.2). By the principle of superposition then $C_1e^{r_1t} + C_2e^{r_2t}$ is also a solution of (2.2) for all choices of the constants $C_1, C_2,$ and we will refer to this as the general homogeneous solution. Then the problem of finding solutions for (2.2) is reduced to the problem of finding the roots of the **auxiliary equation**

$$ar^2 + br + c = 0 \qquad (2.3)$$

Examples

1) Consider the equation

$$y''(t) + 2y'(t) - 3y(t) = 0.$$

The auxiliary equation is

$$r^2 + 2r - 3 = 0$$

with roots $r = 1, -3.$ Then $y_1(t) = e^t$ and $y_2(t) = e^{-3t}$ are both solutions for the differential equation. The function $y(t) = C_1e^t + C_2e^{-3t}$ is the general homogeneous solution.

2. Consider the equation

$$y''(t) + 36y(t) = 0.$$

The auxiliary equation in this case is

$$r^2 + 36 = 0$$

with imaginary roots $r = 6i,\ -6i.$ Then $y_1(t) = e^{i6t}$ and $y_2(t) = e^{-i6t}$ are both solutions for the differential equation. However, these solutions are both complex valued functions and we generally prefer a real valued solution to an equation with real coefficients. If we recall the definitions of the sine and cosine functions

$$\sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad \text{and} \qquad \cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2},$$

then we see that we can form new solutions to the differential equation as follows

$$y_3(t) = \frac{y_1(t) + y_2(t)}{2} = \cos(6t)$$

$$\text{and} \qquad y_4(t) = \frac{y_1(t) - y_2(t)}{2i} = \sin(6t).$$

The functions $y_3(t)$ and $y_4(t)$ are two new solutions to the differential equation and $y(t) = C_1\cos(6t) + C_2\sin(6t)$ is the general homogeneous real valued solution for all choices of $C_1, C_2.$

3. Consider the equation

$$y''(t) + 2y'(t) + 36y(t) = 0.$$

The auxiliary equation in this case is

$$r^2 + 2r + 36 = 0,$$

and its roots are the complex conjugate pair, $r_\pm = -1 \pm i\sqrt{35}$. Then

$$y_1(t) = e^{(-1+i\sqrt{35})t} = e^{-t}e^{\left(i\sqrt{35}\right)t}$$

$$\text{and} \quad y_2(t) = e^{(-1-i\sqrt{35})t} = e^{-t}e^{\left(-i\sqrt{35}\right)t}$$

are both solutions for the differential equation. In this case, forming the combinations

$$y_3(t) = \frac{y_1(t) + y_2(t)}{2} \qquad \text{and} \qquad y_4(t) = \frac{y_1(t) - y_2(t)}{2i},$$

leads to

$$y_3(t) = e^{-t}\cos\left(t\sqrt{35}\right) \qquad \text{and} \qquad y_4(t) = e^{-t}\sin\left(t\sqrt{35}\right).$$

Evidently, when the roots of the auxiliary equation are the complex conjugate pair, $r = \alpha \pm i\beta$, then the corresponding real valued solutions of the differential equation are

$$y_3(t) = e^{\alpha t}\cos(\beta t) \qquad \text{and} \qquad y_4(t) = e^{\alpha t}\sin(\beta t),$$

and $y(t) = e^{\alpha t}[C_1\cos(\beta t) + C_2\sin(\beta t)]$ is the general homogeneous solution.

4. Finally, consider the equation

$$y''(t) + 2y'(t) + y(t) = 0.$$

The auxiliary equation in this case is, $r^2 + 2r + 1 = 0$, which has repeated roots, $r = -1, -1$. Then one solution of the differential equation is the function $y(t) = e^{--t}$, but we expect a second solution. To see what it should be, we note that substituting the guess e^{rt} into the differential equation (1.5) led to

$$L[e^{rt}] = P(r)\,e^{rt} = 0, \quad \text{where} \quad P(r) = ar^2 + br + c.$$

Then we observed that e^{rt} solves the differential equation if r is a root of the auxiliary equation, $P(r) = 0$. When the auxiliary equation is $P(r) = r^2 + 2r + 1 = (r+1)^2 = 0$, we have not only $P(1) = 0$, we have $P'(1) = 0$ as well. That is, when a quadratic equation has a double root at $r = r_0$ then we have $P(r_0) = P'(r_0) = 0$. Then notice that since

$$\frac{d}{dr}L[e^{rt}] = \frac{d}{dr}[P(r)\,e^{rt}]$$

and

$$\frac{d}{dr}L[e^{rt}] = L\left[\frac{d}{dr}e^{rt}\right] = L[t\,e^{rt}],$$

$$\frac{d}{dr}[P(r)\,e^{rt}] = P'(r)e^{rt} + P(r)\,re^{rt},$$

it follows that,

$$L[t\,e^{r_0 t}] = P'(r_0)e^{r_0 t} + P(r_0)\,r_0 e^{r_0 t} = 0.$$

That is, when the quadratic auxiliary equation has a double root at $r = r_0$, then two solutions of the differential equation are the functions,

$$y_1(t) = e^{r_0 t} \qquad \text{and} \qquad y_2(t) = t\,e^{r_0 t}$$

and then the function $y(t) = C_1 e^{r_0 t} + C_2\,te^{r_0 t}$ is the general homogeneous solution for the differential equation.■

In each of the previous examples, the equation had two solutions, and in each case these two solutions were linearly independent. The result Theorem *.* asserts that every linear second order homogeneous differential equation has exactly two linearly independent solutions. Note that in examples 2 and 3 we found four solutions for the homogeneous equations there but in both cases, there are only two linearly independent solutions. The remaining two solutions can be expressed as linear combinations of the other two solutions. In the four examples above, we have found two linearly independent real valued solutions for the homogeneous differential equation in the example. We can summarize what we found in these examples in the following table.

Roots of the aux eqn	**Lin.Indep. solns of homog ODE**
distinct real roots r_1, r_2	$y_1(t) = e^{r_1 t}, \quad y_2(t) = e^{r_2 t}$
imaginary pair $r = \pm i\omega$	$y_1(t) = \cos(\omega t), \quad y_2(t) = \sin(\omega t)$
conjugate pair $r = \alpha \pm i\omega$	$y_1(t) = e^{\alpha t}\cos(\omega t), \; y_2(t) = e^{\alpha t}\sin(\omega t)$
double real root r_1, r_1	$y_1(t) = e^{r_1 t}, \quad y_2(t) = t e^{r_1 t}$

If $y_1(t) \textit{ and } y_2(t)$ are linearly independent solutions of the homogeneous equation (2.2), then the **general solution** of (2.2) is given by $y(t) = C_1 y_1(t) + C_2\, y_2(t)$. The general solution contains two arbitrary constants and we say, therefore, that the general solution of a second order homogeneous ODE is a 2-parameter family of solutions. The constants C_1 and C_2 are the parameters and they provide the flexibility to satisfy, not just one, but two initial conditions. This is what is required if we are to solve initial value problems of the form

$$ay''(t) + by'(t) + cy(t) = 0, \qquad y(t_0) = A_0, \quad y'(t_0) = A_1.$$

Once the general solution $y(t) = C_1 y_1(t) + C_2\, y_2(t)$ has been found, we simply write

$$y(t_0) = C_1 y_1(t_0) + C_2\, y_2(t_0) = A_0$$
$$y'(t_0) = C_1 y_1'(t_0) + C_2\, y_2'(t_0) = A_1,$$

and we solve this set of two equations for the two unknowns C_1 and C_2. The resulting function with these two values for C_1 and C_2 is then the unique solution to the initial value problem.

Example

Consider

$$y''(t) + 2y'(t) - 3y(t) = 0, \qquad y(0) = 2, \quad y'(0) = -2.$$

The general solution for the differential equation was found in a previous example to be,

$$y(t) = C_1 e^{t} + C_2 e^{-3t}.$$

Then the initial conditions imply

$$y(0) = C_1 e^0 + C_2 e^0 = \quad C_1 + C_2 \quad = 2$$
$$y'(0) = C_1 e^0 - 3C_2 e^0 = C_1 - 3C_2 = -2.$$

We solve the two equations generated by the initial conditions to get, $C_1 = 1, \quad C_2 = 1.$ Then $y(t) = e^{t} + e^{-3t}$ is the unique solution of the initial value problem.

Clearly solving constant coefficient differential equations of order higher than two can be accomplished in same way, by assuming a solution of the form $y(t) = e^{rt}$. Substituting this into the n-th order differential equation leads to an algebraic equation of degree n for the roots r. In general, such an algebraic equation will have n roots $r_1, \ldots, r_n$ and then the independent solutions to the differential equation are $e^{r_1 t}, \ldots, e^{r_n t}$. Obviously there is the possibility of repeated roots and complex roots but these can be dealt with in the same way as in the order two case.

3. Free Response of a Spring Mass System

As a particular example of a second order homogeneous linear equation with constant coefficients, we are going to consider a system consisting of a mass suspended on an elastic spring. If the spring is unstretched and the mass, m, is attached to the spring, the spring stretches an amount Δ. Since it is the weight of the mass that stretches the spring by this amount, we have the following equation

$$mg = K\Delta,$$

where g denotes the acceleration of gravity. This is just the assertion that the amount of stretch Δ, is proportional to the weight of the mass and the constant K is called the spring constant. This is sometimes referred to as Hooke's law. Gravity exerts a force mg on the mass and the spring exerts an equal and opposite force equal to $K\Delta$ so that the spring-mass system is in a state of equilibrium.

Now we define our coordinate system to have its origin ($x = 0$) at the point where the mass is located when the system is in its equilibrium position, and we will take x to be positive in the downward direction. We now apply Newton's law to the spring-mass system, that is the mass times acceleration equals the sum of the forces acting on the mass. We let $x(t)$ denote the position of the mass at time t and then $x''(t)$ is the acceleration. The forces that act on the mass are the force of gravity and the spring force so the equation expressing Newton's law becomes,

$$\begin{aligned} mx''(t) &= mg - K[x(t) + \Delta] \\ &= -Kx(t). \end{aligned}$$

The equation $mx''(t) = -Kx(t)$ is the equation of so-called "simple harmonic motion" and the system is called a harmonic oscillator. We consider this equation in detail in this section. We mention in passing that this equation and the modified versions we consider next, occur in numerous other physical settings. In particular, LCR electric circuits lead to these same differential equations.

Suppose that the mass is subject to the additional effect of viscous damping, meaning that the mass experiences a force which is opposite to the motion and proportional to the velocity of the mass. Then a mathematical model for the motion of the damped spring mass system is obtained by equating the mass times the acceleration to the sum of the spring force and the viscous friction force,

$$\begin{aligned} m\,a + mg &= F_{Spring} + F_{Friction} \\ m\,x''(t) + mg &= -K(x(t) - D) - c\,x'(t). \end{aligned}$$

The constant c is assumed to be positive so the force $-cx'(t)$ is acting in opposition to the motion of the mass.Then the mathematical model for a damped spring-mass system with no forcing and initial conditions that correspond to an initial displacement of the mass by the amount x_0, and an initial velocity of v_0 for the mass is as follows,

$$m\,x''(t) = -K\,x(t) - c\,x'(t) \qquad x(0) = x_0, \quad x'(0) = v_0.$$

There are four cases we will consider:

1) **The undamped case**- $c = 0$
In this case where there is no damping, the differential equation becomes $mx''(t) + Kx(t) = 0$. The characteristic equation is then, $mr^2 + K = 0$, and its roots are the imaginary pair $r = \pm i\Omega$, where $\Omega = \sqrt{\frac{K}{m}}$ denotes the so called natural frequency of the system.

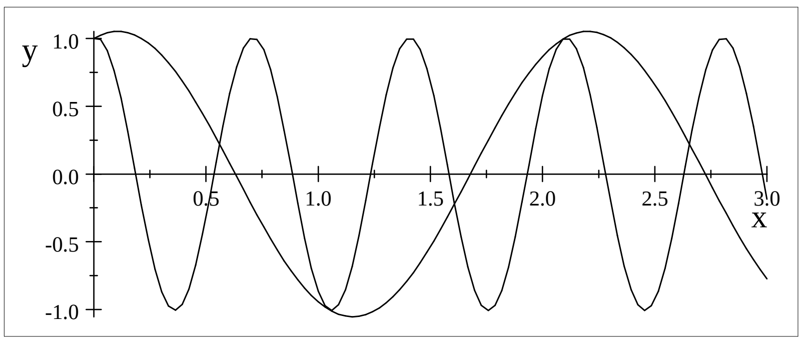
The real valued independent solutions in this case are, $x_1(t) = \cos \Omega t$ and $x_2(t) = \sin \Omega t$, and the general solution is then

$$x(t) = C_1 \cos \Omega t + C_2 \sin \Omega t.$$

The initial conditions imply, $x(0) = C_1 = x_0$, and $x'(0) = \Omega C_2 = v_0$ hence the unique solution to the initial value problem is

$$x(t) = x_0 \cos \Omega t + \frac{v_0}{\Omega} \sin \Omega t.$$

This is a solution whose amplitude does not decrease with time and whose frequency, Ω, increases as the stiffness K of the spring increases and decreases as the mass m increases.



$x_0 = v_0 = 1$ and $\Omega = 3$ and 9

2) **The Damped Case** $c > 0$
When damping is present, the differential equation, $mx''(t) + cx'(t) + Kx(t) = 0$ has as its characteristic equation, $mr^2 + cr + K = 0$. Then the roots are a complex conjugate pair given by

$$r_{1,2} = \frac{-c \pm \sqrt{c^2 - 4Km}}{2m}$$

There are three distinct cases that can occur, each with its own characteristic solution behavior:

(a) **the underdamped case** $0 < c^2 < 4Km$
In this case the roots are a complex conjugate pair $r_{1,2} = -\alpha \pm i\omega$ where $\alpha = \frac{c}{2m}$ and

$$\begin{aligned} \omega &= \frac{\sqrt{4Km - c^2}}{2m} \\ &= \sqrt{\frac{K}{m}} \sqrt{1 - \frac{c^2}{4mK}} \\ &= \Omega\sqrt{1 - \rho^2} < \Omega. \end{aligned}$$

Here we have introduced the constant $\rho = \frac{c}{2\sqrt{mK}}$, which is a measure of the damping called the "damping ratio". Then $0 < c^2 < 4Km$ is the same as $0 < \rho^2 < 1$.

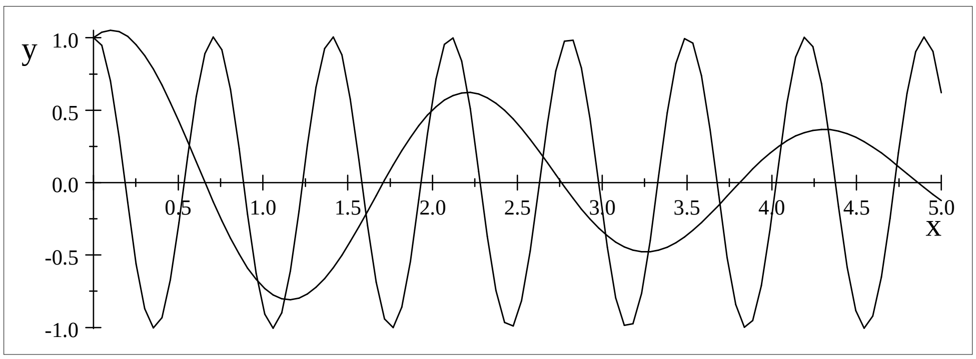
Note that one effect of the damping is to reduce the frequency at which the system oscillates. The frequency of oscillation for the damped system decreases from the natural frequency Ω, to a lower frequency, ω. In this case two independent real valued solutions are $x_1(t) = e^{-\alpha t}\cos\omega t$ and $x_2(t) = e^{-\alpha t}\sin\omega t$, and the general solution is

$$x(t) = C_1 e^{-\alpha t}\cos\omega t + C_2 e^{-\alpha t}\sin\omega t.$$

Then $x(0) = C_1 = x_0$ and $x'(0) = -\alpha C_1 + \omega C_2 = v_0$, so the unique solution to the initial value problem is

$$x(t) = x_0 e^{-\alpha t}\cos\omega t + \frac{v_0 + \alpha x_0}{\omega} e^{-\alpha t}\sin\omega t.$$

Since $\alpha = -\frac{c}{2m} < 0$, this is an oscillating solution whose amplitude decreases exponentially as time increases. The following figure shows the undamped and underdamped solutions plotted on the same axes.



$$m = 1 \quad K = 9 \quad c = 0.5$$

It is clear from the figure that each amplitude peak of the damped solution is less than the previous peak so the amplitude is steadily decreasing. It is also clear that the system oscillates more slowly than when there is no damping.

(b) **the critically damped case** $c^2 = 4Km$ or $\rho = 1$

As the damping is increased, the damped frequency, ω, decreases until finally reaching zero when $c^2 = 4Km$. At this point, the system no longer oscillates at all and the character of the solution changes. In this case, the roots of the characteristic equation are real and equal with the value $r_{1,2} = -\alpha = -\frac{c}{2m}$. Then $x_1(t) = e^{-\alpha t}$ *and* $x_2(t) = te^{-\alpha t}$ are two linearly independent solutions and the general solution is $x(t) = C_1 e^{-\alpha t} + C_2 te^{-\alpha t}$. The initial conditions are satisfied by the following unique solution

$$x(t) = x_0 e^{-\alpha t} + (v_0 + \alpha x_0)\, te^{-\alpha t}.$$

This solution decays to zero without oscillating.

(c) **the overdamped case** $c^2 > 4Km$ or $\rho > 1$

When the damping increases beyond the critical point $c^2 = 4Km$, the characteristic equation has distinct real roots

$$
\begin{aligned}
r_{1,2} &= \frac{-c \pm \sqrt{c^2 - 4Km}}{2m} \\
&= -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{K}{m}} \\
&= -\alpha \pm \beta,
\end{aligned}
$$

where $\qquad \beta = \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{K}{m}} < \alpha$

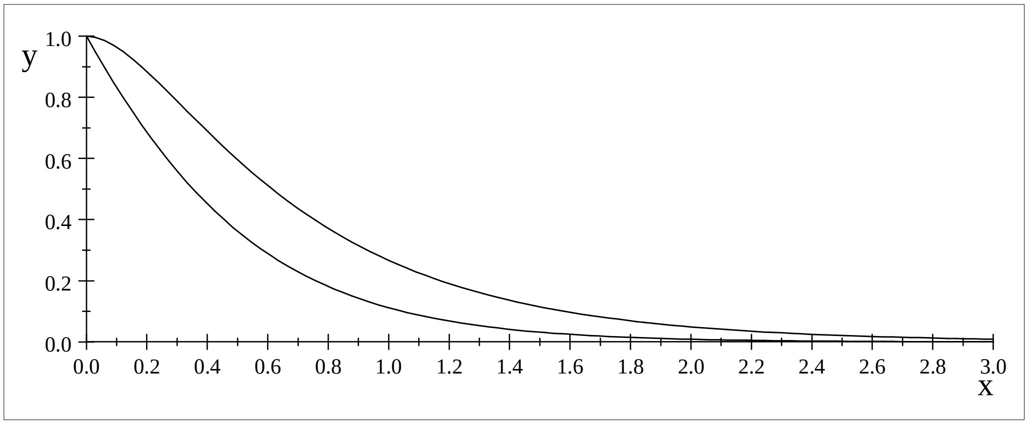
both of which are negative since $-\alpha - \beta < -\alpha < -\alpha + \beta < 0.$ This leads to the general solution

$$x(t) = C_1 e^{-(\alpha-\beta)t} + C_2\, e^{-(\alpha+\beta)t}$$

and the unique solution that satisfies the initial conditions is given by

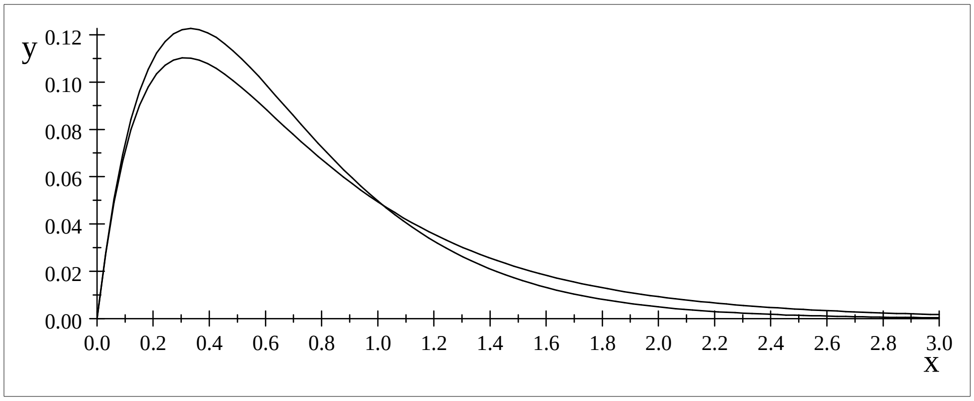
$$x(t) = \frac{v_0 + (\alpha+\beta)x_0}{2\beta}\, e^{-(\alpha-\beta)t} - \frac{v_0 + (\alpha-\beta)x_0}{2\beta}\, e^{-(\alpha+\beta)t}$$

This solution, like the critically damped solution, decays to zero without oscillating. Note that if $x_0 > 0$ *and* $v_0 = 0,$ then the overdamped solution decays to zero less rapidly than the critically damped solution. This is shown in the figure below where both solutions are plotted on the same axes. This is because when the damping is increased beyond the critical value, not only is the oscillation suppressed, the motion of the mass is further impeded so that a mass released from rest takes longer to return to the equilibrium position. Heavy doors are often equipped with a spring device that pulls the door shut but the motion is sufficiently damped that the door can not slam. For this to work properly, the damping must equal or exceed the critical value.



Overdamped and Critically Damped Solutions

If the initial conditions are changed to $x_0 = 0$ *and* $v_0 > 0,$ then both solutions begin from the equilibrium position, $x = 0,$ and rise to a maximum displacement before decreasing back to $x = 0.$ This is illustrated in the figure below. The critically damped solution reaches a higher maximum deflection than does the overdamped solution and therefore takes a longer time to decay to zero. This set of initial conditions is a reasonable approximation of the situation that occurs when a shock absorber is actuated by a car hitting a bump. The shock absorber is compressed by the bump but returns to the uncompressed state without oscillating as long as the damping exceeds the critical value. If the damping is less than the critical value, then the car will bob up and down after hitting a bump. It is time then to replace the shock absorbers.



Initial Conditions $x_0 = 0,\ v_0 > 0$

Decay of the Total Energy

The unforced damped spring mass system is governed by the differential equation

$$mx''(t) + cx'(t) + Kx(t) = 0$$

hence if we define the total energy $E(t)$ in the system to be the sum of kinetic energy and potential energy, then

$$E(t) = \frac{1}{2}mx'(t)^2 + \frac{1}{2}Kx(t)^2$$

then

$$\begin{aligned} E'(t) &= \frac{1}{2}m2x'(t)x''(t) + \frac{1}{2}K2x(t)x'(t) \\ &= x'(t)[mx''(t) + Kx(t)] = -cx'(t)^2 \end{aligned}$$

If $c = 0$ (the undamped case) then $E'(t) = 0,$ so the the total energy is constant; the energy only changes from kinetic to potential and back again. At the instant when the mass reaches its maximum deflection, the velocity is zero and the energy is all potential energy stored in the spring. At the instant when the mass passes through the equilibrium position at $x = 0,$ the velocity is maximal and the energy is all kinetic since the potential energy is zero when the spring is in its equilibrium state. If $c > 0,$ then $E'(t) = -cx'(t)^2 < 0$ so the energy is decreasing, which is why the friction is referred to as an energy dissipating mechanism. The rate at which energy is dissipated increases as the damping increases.

The equation of the undamped and damped harmonic oscillator applies to other physical systems besides the spring-mass system. These include electric circuits and bouyant object floating in a fluid.

4. Forced Response of a Spring Mass System

We are going to consider the response of the spring-mass system when it is subject to a forcing function. This is also called the forced harmonic oscillator. In addition to an elastic system, the equation governs electric circuits with a voltage input. Since the forced system involves the inhomogeneous ODE, we are going to recall the method of undetermined coefficients as it applies to a linear second order ODE with constant coefficients. This simple method of finding a particular solution for an inhomogeneous equation will be sufficient for the few examples we consider in this section. In the next section we will present a more general method of finding particular solutions to a second order linear ODE.

4.1 The Method of Undetermined Coefficients

Consider the equation

$$L[y(t)] = ay''(t) + by'(t) + cy(t) = f(t).$$

One way of determining a particular solution is by the method of undetermined coefficients. For simple forcing functions $f(t)$ it is often easy to guess the form of the particular solution and to then determine the coefficients in the form from substituting into the equation. A list of forms to be guessed for various simple forcing functions is provided by the following table, with the forcing function $f(t)$ in the left hand column of the table and the appropriate guess for $y_p(t)$ in the right hand column:

$f(t)$	$y_p(t)$
a	A
bt	$A + Bt$
ct^2	$A + Bt + Ct^2$
$a + bt + ct^2$	$A + Bt + Ct^2$
e^{bt}	Ae^{bt}
te^{bt}	$(A + Bt)e^{bt}$
$\sin\Omega t$	$A\sin\Omega t + B\cos\Omega t$
$\cos\Omega t$	$A\sin\Omega t + B\cos\Omega t$

For example, consider

$$y''(t) + 25y(t) = 3\sin 4t$$

According to the table we should suppose

$$y_p(t) = A\sin 4t + B\cos 4t$$

Then

$$y_p'(t) = 4A\cos 4t - 4B\sin 4t$$
$$y_p''(t) = -16A\sin 4t - 16B\cos 4t$$

and

$$\begin{aligned} y_p''(t) + 25y_p(t) &= -16A\sin 4t - 16B\cos 4t + 25(A\sin 4t + B\cos 4t) \\ &= 9A\sin 4t + 9B\cos 4t. \end{aligned}$$

The original differential equation implies that $9A\sin 4t + 9B\cos 4t = 3\sin 4t,$ from which it follows that $A = \frac{1}{3}$ and $B = 0$.

This was a particularly simple example. A more complicated example arises if we consider

$$y''(t) + 2y'(t) + 25y(t) = 3\sin 4t$$

If we make the same guess for $y_p(t)$, then

$$
\begin{aligned}
y''(t) + 2y'(t) + 25y(t) &= \\
&= -16A\sin 4t - 16B\cos 4t + 2(4A\cos 4t - 4B\sin 4t) + 25(A\sin 4t + B\cos 4t) \\
&= (-16A - 8B + 25A)\sin 4t + (-16B + 8A + 25B)\cos 4t \\
&= (9A - 8B)\sin 4t + (9B + 8A)\cos 4t
\end{aligned}
$$

Then it follows from the original differential equation that

$$9A - 8B = 3$$

$$\textit{and} \qquad 9B + 8A = 0$$

i.e.,

$$A = \tfrac{27}{145} \qquad B = -\tfrac{24}{145}$$

$$\textit{and} \qquad y_p(t) = \tfrac{27}{145}\sin 4t - \tfrac{24}{145}\cos 4t$$

Finally, consider the example

$$y''(t) + 2y'(t) + 25y(t) = 3te^{-t}$$

We guess that $\quad y_p(t) = (A + Bt)e^{-t}$

Then

$$y_p'(t) = Be^{-t} - (A + Bt)e^{-t} = (B - A - Bt)e^{-t}$$

$$y_p''(t) = -2Be^{-t} + (A + Bt)e^{-t} = (A - 2B + Bt)e^{-t}$$

and so the equation implies

$$y_p''(t) + 2y_p'(t) + 25y_p(t) = [(A - 2B + Bt) + 2(B - A - Bt) + 25(A + Bt)]e^{-t}$$

$$= [24A + 24Bt]e^{-t} = 3te^{-t}$$

Then $A = 0$ and $B = \frac{1}{8}$, so $\quad y_p(t) = \frac{1}{8}te^{-t}$.

Now we consider some applications related to the spring-mass system that lead to problems with simple forcing functions.

4.2. The Undamped Forced Response

Consider a periodically forced, undamped spring mass system, starting from rest in a relaxed state. Then governing equation is

$$
\begin{aligned}
m x''(t) + K x(t) &= F \cos \omega t, \\
x(0) &= 0, \text{ (system is initially relaxed)} \\
x'(0) &= 0 \text{ (system is initially at rest)}
\end{aligned}
$$

Let $\Omega = \sqrt{\frac{K}{m}}$ and suppose $\Omega \neq \omega$. The homogeneous solution is known to be

$$x(t) = C_1 \cos \Omega t + C_2 \sin \Omega t,$$

and if we suppose the particular solution is of the form

$$x_p(t) = A \cos \omega t + B \sin \omega t,$$

then

$$(-m\omega^2 A + KA) \cos \omega t + (-m\omega^2 B + KB) \sin \omega t = F \cos \omega t.$$

Equating coefficients leads to

$$
\begin{aligned}
A &= \frac{F}{m} \frac{1}{\frac{K}{m} - \omega^2} \\
&= \frac{F}{m} \frac{1}{\Omega^2 - \omega^2}, \\
B &= 0.
\end{aligned}
$$

Then

$$x_p(t) = \frac{F}{m} \frac{1}{\Omega^2 - \omega^2} \cos \omega t$$

and the general solution is

$$x(t) = C_1 \cos \Omega t + C_2 \sin \Omega t + \frac{F}{m} \frac{1}{\Omega^2 - \omega^2} \cos \omega t.$$

The initial conditions imply that

$$
\begin{aligned}
x(0) &= C_1 + \frac{F}{m} \frac{1}{\Omega^2 - \omega^2} = 0, \\
x'(0) &= \Omega C_2 = 0,
\end{aligned}
$$

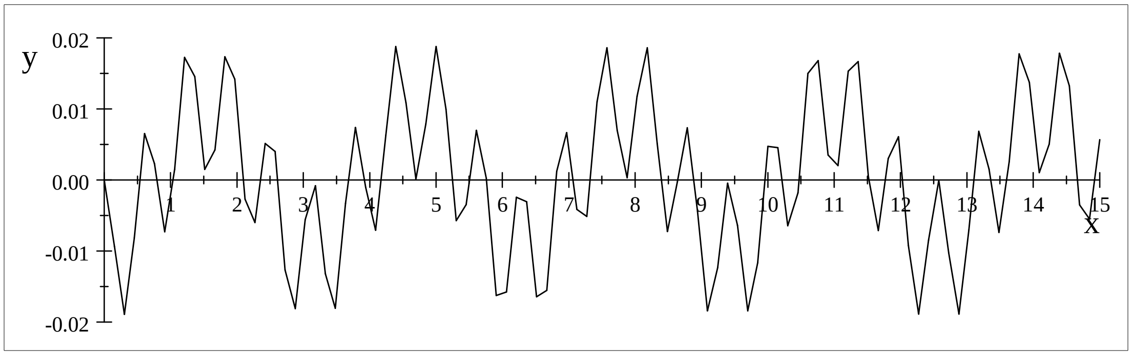
so

$$
\begin{aligned}
C_1 &= -\frac{F}{m} \frac{1}{\Omega^2 - \omega^2}, \\
\textit{and} \qquad C_2 &= 0.
\end{aligned}
$$

Then

$$x(t) = \frac{F}{m} \frac{1}{\Omega^2 - \omega^2} [\cos \omega t - \cos \Omega t].$$

The solution may look like the following figure



$$\Omega = 10 \text{ and } \omega = 2$$

However, when ω and Ω are close, the behavior is much different. In order to better see how the solution behaves in this case, let

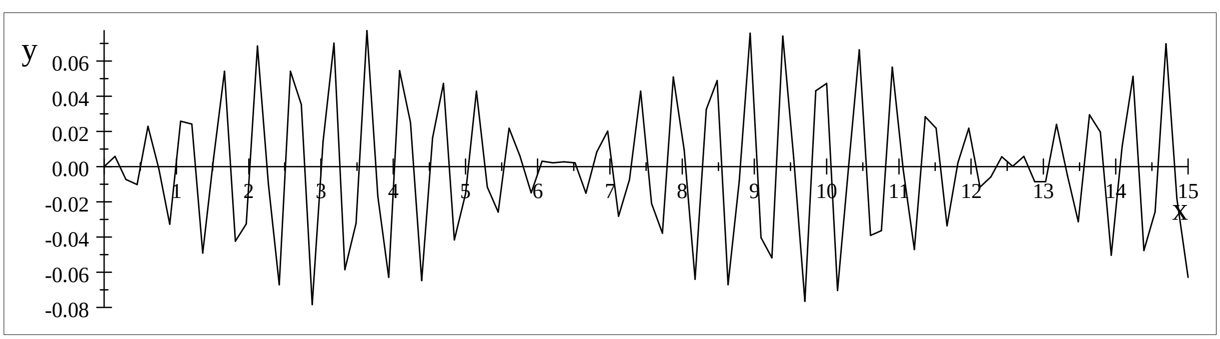
$$\phi = \frac{1}{2}(\Omega + \omega)$$
$$\sigma = \frac{1}{2}(\Omega - \omega)$$

Here ϕ is called the "fast frequency" and is equal to the average of two nearly equal frequencies, Ω and ω. On the other hand, σ is very small and is called the "slow frequency". Then $\Omega = \phi + \sigma$ *and* $\omega = \phi - \sigma$, and the trigonometric identity for the cosine of a sum or difference leads to

$$\cos \Omega t = \cos(\phi + \sigma)t = \cos \phi t \cos \sigma t - \sin \phi t \sin \sigma t$$
$$\cos \omega t = \cos(\phi - \sigma)t = \cos \phi t \cos \sigma t + \sin \phi t \sin \sigma t$$

Then $\cos \omega t - \cos \Omega t = 2 \sin \phi t \sin \sigma t.$ and $x(t)$ can be expressed as

$$x(t) = \frac{F}{m} \frac{2}{\Omega^2 - \omega^2} \sin \phi t \sin \sigma t$$



$$\Omega = 13 \quad \omega = 12$$

This plot shows how $x(t)$ looks. The enveloping sine curve is the ”slow frequency” $\sin \sigma t = \sin \frac{1}{2}(\Omega - \omega)t$, and the ”fast frequency”, $\sin \phi t = \sin \frac{1}{2}(\Omega + \omega)t$, oscillates inside this bound. This periodic increase and decrease in amplitude is sometimes referred to as the phenomenon of ”beats”.

Returning to the original way of writing the solution,

$$x(t) = \frac{F}{m}\frac{1}{\Omega^2 - \omega^2}[\cos\omega t - \cos\Omega t].$$

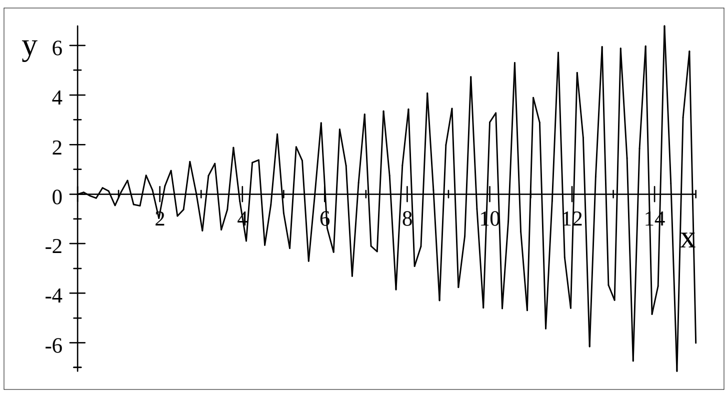
we can examine what happens as the forcing frequency $\omega,$ approaches the natural frequency $\Omega,$ of the system. We take the limit as $\omega \to \Omega,$ and find

$$\lim_{\omega\to\Omega}\frac{\cos\omega t - \cos\Omega t}{\Omega^2 - \omega^2} = \frac{t\sin\Omega t}{2\Omega}$$

where we used L'Hopital's rule to evaluate the limit. Then, as ω tends to $\Omega,$ the solution tends to the limit,

$$x(t) = \frac{F}{2m\Omega}\, t\sin\Omega t$$

This is the solution when the forcing frequency is the same as the natural frequency of the spring mass system, $\Omega = \sqrt{\frac{K}{m}}$. This solution, known as the **resonant response**, has steadily increasing amplitude as seen in the following plot,



The implication of this solution is that in a system with negligible damping, the amplitude of the response would continue to grow until, at some point, the elastic limit of the system is exceeded and the system breaks. We will see now how the addition of damping changes the response.

4.3. The Damped Forced Response

Consider a periodically forced, damped spring mass system, starting from rest in a relaxed state. Then governing equation in this case is

$$mx''(t) + cx'(t) + Kx(t) = F\cos\omega t,$$

$$x(0) = 0, \text{ (system is initially relaxed)}$$

$$x'(0) = 0 \text{ (system is initially at rest)}$$

As in the previous section, $\Omega = \sqrt{\frac{K}{m}}$ and we suppose first that $\Omega \neq \omega.$

The homogeneous solution is given by

$$x_H(t) = e^{\beta t}(C_1 \cos \omega_d t + C_2 \sin \omega_d t)$$

where

$$\beta = -\frac{c}{2m} \qquad \text{and} \qquad \omega_d = \sqrt{\Omega^2 - \left(\frac{c}{2m}\right)^2} = \Omega\sqrt{1-\rho^2}.$$

If we suppose the particular solution is of the form

$$x_p(t) = A\cos\omega t + B\sin\omega t,$$

then

$$x_p'(t) = \omega B\cos\omega t - \omega A\sin\omega t$$

$$x_p''(t) = -\omega^2 A\cos\omega t - \omega^2 B\sin\omega t$$

and $\quad m x_p''(t) + c x_p'(t) + K x_p(t)$

$$= (KA - m\omega^2 A + c\omega B)\cos\omega t + (KB - m\omega^2 B - c\omega A)\sin\omega t = F\cos\omega t.$$

Equating coefficients on the two sides of this last equation, we obtain

$$(K - m\omega^2)A + c\omega B = F$$

$$\text{and} \qquad -c\omega A + (K - m\omega^2)B = 0.$$

This leads to

$$A = \frac{K - m\omega^2}{(K-m\omega^2)^2 + (c\omega)^2}F,$$

$$B = \frac{c\omega}{(K-m\omega^2)^2 + (c\omega)^2}F$$

and so the particular solution is,

$$x_p(t) = \frac{F}{(K-m\omega^2)^2 + (c\omega)^2}[(K - m\omega^2)\cos\omega t + (c\omega)\sin\omega t],$$

$$= \frac{F}{\sqrt{(K-m\omega^2)^2 + (c\omega)^2}}[\cos\theta\cos\omega t + \sin\theta\sin\omega t].$$

We have written the solution in terms of the phase shift θ, where

$$\cos\theta = \frac{K - m\omega^2}{\sqrt{(K-m\omega^2)^2 + (c\omega)^2}},$$

$$\sin\theta = \frac{c\omega}{\sqrt{(K-m\omega^2)^2 + (c\omega)^2}},$$

$$\text{i.e.,} \qquad \theta = Tan^{-1}\left(\frac{c\omega}{K - m\omega^2}\right).$$

Then the particular solution can be written as,

$$x_p(t) = \frac{F}{\sqrt{(K-m\omega^2)^2 + (c\omega)^2}}\cos(\omega t - \theta)$$

and the general solution is,

$$x(t) = e^{-\beta t}(C_1\cos\omega_d t + C_2\sin\omega_d t) + \frac{F}{\sqrt{(K-m\omega^2)^2 + (c\omega)^2}}\cos(\omega t - \theta)$$

Now the initial conditions imply

$$C_1 + \frac{F}{\sqrt{(K - m\omega^2)^2 + (c\omega)^2}} \cos\theta = 0$$

$$-\beta C_1 + \omega_d C_2 + \frac{F}{\sqrt{(K - m\omega^2)^2 + (c\omega)^2}} \omega \sin\theta = 0$$

and this leads to the result

$$C_1 = \frac{-F}{\sqrt{(K - m\omega^2)^2 + (c\omega)^2}} \cos\theta$$

$$C_2 = \frac{-F}{\sqrt{(K - m\omega^2)^2 + (c\omega)^2}} \left[\frac{\beta\cos\theta + \omega\sin\theta}{\omega_d}\right].$$

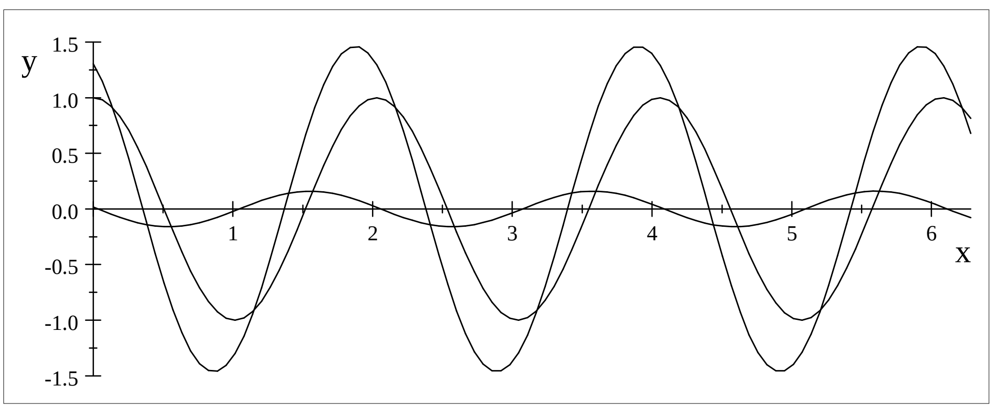
Now the unique solution to the initial value problem is

$$x(t) = \frac{F}{\sqrt{(K - m\omega^2)^2 + (c\omega)^2}} \left[\cos(\omega t - \theta) - e^{-\beta t}\left(\cos\theta\cos\omega_d t + \left(\frac{\beta\cos\theta + \omega\sin\theta}{\omega_d}\right)\sin\omega_d t\right)\right]$$

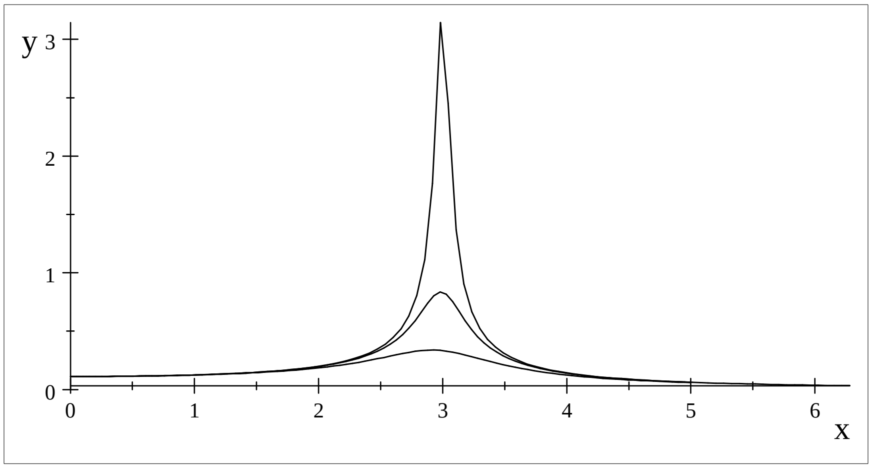
Clearly the steady state part of the solution is given by

$$x_{ss}(t) = \frac{F}{\sqrt{(K - m\omega^2)^2 + (c\omega)^2}} \cos(\omega t - \theta)$$

The following picture shows the steady state plotted on the same axes as the input. The input function is the curve having amplitude equal to 1, (we are taking $F = 1$ and $\omega = 3.1$ for convenience). The curve that starts at about 1.4 is the steady state response when $K = 9$, $m = 1$ and $c = 0.1$. The magnitude of the response curve is about one and a half times that of the input and there is very little phase shift. The response curve that starts from nearly 0 corresponds to a larger value for c, namely $c = 2$. This leads to a response with magnitude about one tenth of the input amplitude and there is a more significant phase shift. .



The reason that the steady state output when $c = 2$ has amplitude less than the input amplitude while the response when $c = 0.1$ has amplitude greater than the amplitude of the input is illustrated in the following figure.



Amplification Factor vs ω for $c = .1, .4, 1$ with $\Omega = 3$

Here we have plotted the amplification factor,

$$\frac{|x_{ss}(t)|}{F} = \frac{1}{\sqrt{(K - m\omega^2)^2 + (c\omega)^2}}$$

versus ω for three different values of damping, c. The highest amplification occurs for the smallest value of damping. For $c = .1$, the amplification is around 2.5 at an input frequency of $\omega = 3.1$, while the amplification is around0.2 at this same frequency when $c = 1$. The middle curve, where the max amplification is about $.8$, corresponds to a damping of $c = 0.4$.

Note that when $c = 0$, the amplification becomes infinite at $\omega = \Omega$. Thus the addition of damping modifies the behavior at resonance from there being an infinite (and physically meaningless) magnification to where the magnification is finite (and hence physically reasonable).

5. Variation of Parameters

In this section we will see how the variation of parameters method that we encountered in finding particular solutions for first order equations can be modified to work for equations of order two. We continue to restrict our attention to linear equations with constant coefficients.

Consider the initial value problem

$$L[y(t)] = y''(t) + ay'(t) + by(t) = f(t), \qquad y(0) = y'(0) = 0.$$

Suppose $y_1(t)$ *and* $y_2(t)$ are two linearly independent solutions of the homogeneous equation. Then the general solution of the homogeneous equation is given by

$$y_H(t) = C_1 y_1(t) + C_2 y_2(t).$$

Motivated by the variation of parameters technique that we used in finding particular solutions to first order linear ODE's, we consider a solution for the inhomogeneous second order equation that has the form

$$y_p(t) = C_1(t) y_1(t) + C_2(t) y_2(t) \qquad (1)$$

where the unknown functions $C_1(t)$ *and* $C_2(t)$ remain to be found. In preparation for substituting this guess for the particular solution into the inhomgeneous equation, we compute

$$y_p'(t) = C_1'(t)y_1(t) + C_2'(t)y_2(t) + C_1(t)y_1'(t) + C_2(t)y_2'(t).$$

Before computing $y_p''(t)$, first we will impose the following condition on $C_1(t)$ *and* $C_2(t)$,

$$C_1'(t)y_1(t) + C_2'(t)y_2(t) = 0. \qquad (2)$$

Then,

$$y_p'(t) = C_1(t)y_1'(t) + C_2(t)y_2'(t),$$

and $$y_p''(t) = C_1'(t)y_1'(t) + C_2'(t)y_2'(t) + C_1(t)y_1''(t) + C_2(t)y_2''(t)$$

Now we have

$$\begin{aligned} y_p''(t) + ay_p'(t) + by_p(t) = {} & C_1'(t)y_1'(t) + C_2'(t)y_2'(t) + C_1(t)y_1''(t) + C_2(t)y_2''(t) \\ & + a(C_1(t)y_1'(t) + C_2(t)y_2'(t)) \\ & + b(C_1(t)y_1(t) + C_2(t)y_2(t)) = \end{aligned}$$

$$\begin{aligned} & = C_1'(t)y_1'(t) + C_2'(t)y_2'(t) + C_1(t)[y_1''(t) + ay_1'(t) + by_1(t)] \\ & \qquad + C_2(t)[y_2''(t) + ay_2'(t) + by_2(t)] \\ & = C_1'(t)y_1'(t) + C_2'(t)y_2'(t) + 0 + 0 = f(t). \end{aligned}$$

That is, substituting our guess (1) into the differential equation, and taking into account the condition, (2), causes the equation to reduce to only the following terms

$$C_1'(t)y_1'(t) + C_2'(t)y_2'(t) = f(t). \qquad (3)$$

The rest of the terms cancel out, as they did in the V of P's for first order equations, because $y_1(t)$ *and* $y_2(t)$ are solutions of the homogeneous equation. Thus the inhomogeneous differential equation imposes only a single condition on the unknown functions $C_1(t)$ *and* $C_2(t)$, this is the equation (3). In order to get a unique pair of functions, we need to impose a second condition on $C_1(t)$ *and* $C_2(t)$. We imposed the condition (2) for the reason that it made the computations shorter, but we could just as well impose other conditions on $C_1(t)$ *and* $C_2(t)$ if we wanted to.

If we use (2) and (3) to determine $C_1(t)$ *and* $C_2(t)$, then

$$C_1'(t)y_1(t) + C_2'(t)y_2(t) = 0. \qquad (2)$$

$$C_1'(t)y_1'(t) + C_2'(t)y_2'(t) = f(t). \qquad (3)$$

and we must solve this set of two equations for $C_1'(t)$ *and* $C_2'(t)$. Solving leads to

$$C_1'(t) = -\frac{y_2(t)}{W[y_1,y_2](t)}f(t), \quad \text{and} \quad C_2'(t) = \frac{y_1(t)}{W[y_1,y_2](t)}f(t).$$

Note that the determinant of the algebraic equations for $C_1'(t)$ *and* $C_2'(t)$ is the Wronskian determinant for the homogeneous solutions.

These equations are for the *derivatives* of $C_1(t)$ *and* $C_2(t)$, so we have to integrate to get the functions. We write this as follows

$$C_1(t) = -\int_0^t \frac{y_2(s)}{W[y_1,y_2](s)}f(s)\,ds, \quad \text{and} \quad C_2(t) = \int_0^t \frac{y_1(s)}{W[y_1,y_2](s)}f(s)\,ds,$$

where the limits of integration are 0 (so that $C_1(0) = 0$ *and* $C_2(0) = 0$ in keeping with the homogeneous initial conditions) and t (so that the result of the integration is a function of t, not just a number). Then

$$
\begin{aligned}
y_p(t) &= C_1(t)y_1(t) + C_2(t)y_2(t) \\
&= y_2(t)\int_0^t \frac{y_1(s)}{W[y_1,y_2](s)}\, f(s)\,ds - y_1(t)\int_0^t \frac{y_2(s)}{W[y_1,y_2](s)}\, f(s)\,ds \\
&= \int_0^t \frac{y_2(t)y_1(s) - y_1(t)y_2(s)}{W[y_1,y_2](s)}\, f(s)\,ds \\
&= \int_0^t Y(t,s)\, f(s)\,ds.
\end{aligned}
$$

Here we have combined the expressions involving $C_1(t)y_1(t)$ *and* $C_2(t)y_2(t)$ to write

$$y_p(t) = \int_0^t Y(t,s)\, f(s)\,ds$$

where

$$Y(t,s) = \frac{y_2(t)y_1(s) - y_1(t)y_2(s)}{W[y_1,y_2](s)}.$$

We have a reason for writing the solution this way. We are solving the equation

$$L[y_p(t)] = f(t)$$

where L denotes the differential operator $L[y(t)] = y''(t) + ay'(t) + by(t)$. Then we have just found that the solution of the equation is given by

$$y_p(t) = K[f(t)] := \int_0^t Y(t,s)\, f(s)\,ds,$$

where K denotes the integral operator defined in the equation above. The implication is that $K = L^{-1}$; i.e., the integral operator K is the inverse of the differential operator L. We will not pursue this idea further but we should at least realize that in solving the inhomogeneous differential equation, we are in fact inverting the differential operator.

Note, incidentally that

$$y_p(0) = \int_0^0 Y(t,s)\, f(s)\,ds = 0$$

and

$$y_p'(t) = Y(t,t)f(t) + \int_0^t \frac{d}{dt}Y(t,s)\, f(s)\,ds$$

so

$$y_p'(0) = Y(0,0)f(0) + \int_0^0 \frac{d}{dt}Y(t,s)\, f(s)\,ds = 0.$$

Evidently, the initial conditions are automatically satisfied because we chose the lower limit in the integral to be the point at which the initial conditions are imposed, namely $t = 0$.

We note in passing that if we have solved the homogeneous equation to find two independent solutions, $y_1(t)$ and $y_2(t)$, then we can form $Y(t,s)$ without going through the variation of parameters procedure simply by writing

$$Y(t,s) = \frac{y_2(t)y_1(s) - y_1(t)y_2(s)}{W[y_1,y_2](s)}$$

and then

$$y_p(t) = \int_0^t Y(t,s)\, f(s)\, ds$$

is the particular solution which satisfies $y_p(0) = y_p'(0) = 0.$ If we have nonzero initial conditions to satisfy, we simply add $C_1 y_1(t) + C_2 y_2(t)$ to the particular solution and solve for C_1 and C_2.

Example

Consider

$$y''(t) + \omega^2 y(t) = f(t), \qquad y(0) = y'(0) = 0,$$

where the forcing function, $f(t),$ is unspecified at this point. Later, we can substitute specific examples for $f(t)$. Then

$$y_H(t) = C_1 \cos \omega t + C_2 \sin \omega t,$$

and we let

$$y_p(t) = C_1(t) \cos \omega t + C_2(t) \sin \omega t.$$

Then

$$C_1'(t) \cos \omega t + C_2'(t) \sin \omega t = 0,$$
$$C_1'(t)(-\omega \sin \omega t) + C_2'(t)\, \omega \cos \omega t = f(t),$$

where the first equation is the arbitrary condition, $(2,)$ that we imposed and the second equation is the consequence of substituting $y_p(t)$ into the differential equation. Then

$$C_1'(t) = -\frac{y_2(t)}{W[y_1,y_2](t)} f(t) = -\frac{1}{\omega} \sin \omega t\, f(t),$$
$$C_2'(t) = \frac{y_1(t)}{W[y_1,y_2](t)} f(t) = \frac{1}{\omega} \cos \omega t\, f(t),$$

where we made use of the fact that,

$$W[y_1,y_2](t) = \begin{vmatrix} \cos \omega t & \sin \omega t \\ -\omega \sin \omega t & \omega \cos \omega t \end{vmatrix} = \omega$$

Then

$$C_1(t) = -\int_0^t \frac{1}{\omega} \sin \omega s\ f(s)\, ds, \quad C_2(t) = \int_0^t \frac{1}{\omega} \cos \omega s\ f(s)\, ds,$$

and

$$\begin{aligned} y_p(t) &= C_2(t) \sin \omega t + C_1(t) \cos \omega t \\ &= \int_0^t \frac{1}{\omega} [\cos \omega s\ \sin \omega t - \sin \omega s \cos \omega t]\, f(s)\, ds, \\ &= \int_0^t \frac{1}{\omega} \sin \omega(t-s)\, f(s)\, ds. \end{aligned}$$

If we are given a specific definition for $f(t)$ then we can compute the integral. For example, if

$$f(t) = \begin{cases} 1 & if \ \ 0 < t < 1 \\ 0 & if \ \ t > 1 \end{cases}$$

then

$$\begin{aligned} y_p(t) &= \int_0^t \frac{1}{\omega} \sin \omega(t-s) \ ds. \qquad if \qquad 0 < t < 1 \\ &= \int_0^1 \frac{1}{\omega} \sin \omega(t-s) \ ds. \qquad if \qquad t > 1. \end{aligned}$$

Integrating, we find

$$y_p(t) = \begin{cases} \frac{1}{\omega^2}(1 - \cos \omega t) & if \quad 0 < t < 1 \\ \frac{1}{\omega^2}(\cos \omega(t-1) - \cos \omega t) & if \quad t > 1 \end{cases}$$

An inhomogeneous equation with a piecewise defined forcing function is handled more easily using variation of parameters than trying to use undetermined coefficients.

Exercises

In each of the following problems, find the homogeneous solution, find a particular solution and find the general solution to the inhomogeneous equation. If there are initial conditions, evaluate the arbitrary constants to find the unique solution to the initial value problem.

1. $y''(t) - 3y'(t) - 4y(t) = e^{-t} \qquad y(0) = 0, \ y'(0) = 2$

2. $y''(t) - 3y'(t) + 2y(t) = e^{t}$

3. $y''(t) - y'(t) + 2y(t) = 2\cos t$

4. $y''(t) - 5y'(t) + 6y(t) = 2\cos 2t \qquad y(0) = 0, \ y'(0) = 2$

5. $y''(t) - 3y'(t) - 4y(t) = 2\sin 4t$

In each of the following problems, find a particular solution for the following equations by using variation of parameters:

6. $y''(t) + 16y(t) = 3 \ \sec 4t$

7. $y''(t) + 12y'(t) - 3y(t) = e^{-t^2}$ you can leave the solution in the form of an integral

8. $y''(t) + 4y'(t) + 3y(t) = \begin{cases} 2 & 0 < t < 2 \\ 0 & otherwise \end{cases}$

6. Laplace Transform

In this section we are going to present a method for finding the unique solution to initial value problems for linear differential equations with constant coefficients. It can be applied to equations of any order but we are going to consider only equations of order one or two. This method finds the unique solution to an initial value problem all in one step without first having to find the homogeneous solution and then a particular solution.

Definition *(Laplace Transform) Let $f(t)$ be defined for $t \geq 0$ and let the Laplace transform of $f(t)$ be defined by,*

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = \hat{f}(s)$$

We use the two notations, $L[f(t)]$ and $\hat{f}(s)$, interchangeably to denote the Laplace transform for $f(t)$.

Examples:

1. $f(t) = 1,\ \forall t \geq 0, \qquad L[1] = \int_0^\infty e^{-st}\, dt = \frac{e^{-st}}{-s}\Big|_{t=0}^{t=\infty} = \frac{1}{s} = \hat{f}(s) \quad \text{for } s > 0$

2. $f(t) = e^{bt},\ \forall t \geq 0, \qquad L[e^{bt}] = \int_0^\infty e^{-(b-s)t}\, dt = \frac{e^{-(b-s)t}}{(s-b)}\Big|_{t=0}^{t=\infty} = \frac{1}{s-b} = \hat{f}(s), \quad \text{for } s > b.$

The Laplace transform is defined for all functions of **exponential type**. That is, any function $f(t)$ which is:

- piecewise continuous;i.e., f has at most finitely many finite jump discontinuities on any interval of finite length
- has exponential growth;i.e., for some positive constants M and k, $|f(t)| \leq Me^{kt} \quad \text{for all } t \geq 0.$.

Properties of the Laplace Transform

The Laplace transform has the following **general properties**:

1. $L[C_1 f(t) + C_2 g(t)] = C_1 \hat{f}(s) + C_2 \hat{g}(s)$
2. $L[f(at)] = \frac{1}{a}\hat{f}\left(\frac{s}{a}\right) \qquad \text{for} \qquad a > 0$
3. $L[f'(t)] = s\hat{f}(s) - f(0)$
4. $L[f''(t)] = s^2\hat{f}(s) - sf(0) - f'(0) \quad etc$
5. $L[tf(t)] = -\hat{f}'(s)$
6. $L[t^2 f(t)] = (-1)^2 \hat{f}''(s) \quad etc$
7. $L[e^{bt} f(t)] = \hat{f}(s-b)$

To see that 3 is true, write

$$\begin{aligned} L[f'(t)] &= \int_0^\infty e^{-st} f'(t)\, dt \\ &= e^{-st} f(t)\Big|_{t=0}^{t=\infty} - \int_0^\infty (-s e^{-st}) f(t)\, dt \\ &= -f(0) + s\int_0^\infty e^{-st} f(t)\, dt \\ &= s\hat{f}(s) - f(0), \end{aligned}$$

where we used integration by parts on the first integral. Since the derivative of $f'(t)$ is $f''(t)$, we can apply (3) to $f'(t)$, to get (4),

$$
\begin{aligned}
L[f''(t)] &= sL[f'(t)] - f'(0) \\
&= s\left\{s\hat{f}(s) - f(0)\right\} - f'(0) \\
&= s^2\hat{f}(s) - sf(0) - f'(0)
\end{aligned}
$$

To show 5, just note that

$$
\begin{aligned}
\frac{d}{ds}\hat{f}(s) &= \frac{d}{ds}\int_0^\infty e^{-st}f(t)\,dt \\
&= \int_0^\infty \frac{d}{ds}e^{-st}f(t)\,dt \\
&= \int_0^\infty -t\,e^{-st}f(t)\,dt \\
&= L[-tf(t)].
\end{aligned}
$$

Property 6 can be obtained by applying property 5 twice; i.e., $L[t^2 f(t)] = L[-t\{-tf(t)\}]$.

Finally, property 7 arises from,

$$
\begin{aligned}
L[e^{bt}f(t)] &= \int_0^\infty e^{bt}f(t)\,e^{-st}\,dt \\
&= \int_0^\infty f(t)\,e^{-(s-b)t}\,dt \\
&= \hat{f}(s-b).
\end{aligned}
$$

More Laplace transform Formulas

The properties 1 through 6 are properties of the Laplace transform, and are valid whatever the function $f(t)$. These are analogous to properties of the derivative like the product and quotient rules for differentiation. These properties can be applied to derive more Laplace transform formulas. The Laplace transform formulas are the Laplace transforms of specific functions and are analogous to differentiation formulas like the derivatives for $x^p, e^x,$ etc. For example,

$$
\begin{aligned}
L[t] &= L[t \cdot 1] = -\frac{d}{ds}\left(\frac{1}{s}\right) = \frac{1}{s^2} \\
L[t^2] &= L[t \cdot t] = -\frac{d}{ds}\left(\frac{1}{s^2}\right) = \frac{2}{s^3} \\
L[t^3] &= L[t \cdot t^2] = -\frac{d}{ds}\left(\frac{2}{s^3}\right) = \frac{6}{s^4}
\end{aligned}
$$

and, more generally,

$$
L[t^n] = \frac{n!}{s^{n+1}},\ n = 0,1,2,\ldots
$$

We can also apply this trick to find the transform of e^{bt},

$$
L[te^{bt}] = -\frac{d}{ds}\left(\frac{1}{s-b}\right) = \frac{1}{(s-b)^2}.
$$

In addition, if $f(t) = \sin t,$ then $f'(t) = \cos t,$ and $f''(t) = -\sin t,$ hence property 4 implies

$$
\begin{aligned}
L[-\sin t] &= s^2\hat{f}(s) - sf(0) - f'(0) \\
&= s^2\hat{f}(s) - 1.
\end{aligned}
$$

But $L[-\sin t] = -\hat{f}(s)$ so

$$-\hat{f}(s) = s^2\hat{f}(s) - 1,$$
$$i.e., \qquad \hat{f}(s) = \frac{1}{s^2+1}.$$

Similarly, if $g(t) = \cos t$, then $g'(t) = -\sin t$, and $g''(t) = -\cos t$, so

$$\begin{aligned} L[-\cos t] &= s^2\hat{g}(s) - sg(0) - g'(0) \\ &= s^2\hat{g}(s) - s \end{aligned}$$

But $L[-\cos t] = -\hat{g}(s)$, so

$$-\hat{g}(s) = s^2\hat{g}(s) - s,$$
$$i.e., \qquad \hat{g}(s) = \frac{s}{s^2+1}$$

Furthermore, property 2 implies that

$$\begin{aligned} L[\sin \omega t] &= \frac{1}{\omega}\,\hat{f}\left(\frac{s}{\omega}\right) \\ &= \frac{1}{\omega}\,\frac{1}{\left(\frac{s}{\omega}\right)^2+1} \\ &= \frac{\omega}{s^2+\omega^2}. \end{aligned}$$

and

$$\begin{aligned} L[\cos \omega t] &= \frac{1}{\omega}\,\hat{g}\left(\frac{s}{\omega}\right) \\ &= \frac{1}{\omega}\,\frac{\frac{s}{\omega}}{\left(\frac{s}{\omega}\right)^2+1} \\ &= \frac{s}{s^2+\omega^2}. \end{aligned}$$

In addition

$$\begin{aligned} L[t\sin\omega t] &= -\frac{d}{ds}\left(\frac{\omega}{s^2+\omega^2}\right) \\ &= \frac{2\omega s}{(s^2+\omega^2)^2} \end{aligned}$$

and

$$\begin{aligned} L[t\cos\omega t] &= -\frac{d}{ds}\left(\frac{s}{s^2+\omega^2}\right) \\ &= \frac{s^2-\omega^2}{(s^2+\omega^2)^2}. \end{aligned}$$

We can use property 7 together with the results

$$L[\sin \omega t] = \frac{\omega}{s^2+\omega^2}$$
$$\text{and} \qquad L[\cos \omega t] = \frac{s}{s^2+\omega^2}$$

to derive the formulas

$$L[e^{bt}\sin\omega t] = \frac{\omega}{(s-b)^2+\omega^2}$$
$$\textit{and} \qquad L[e^{bt}\cos\omega t] = \frac{s-b}{(s-b)^2+\omega^2}$$

Application of the Transform

The Laplace transform can be used to solve initial value problems for linear differential equations having constant coefficients. For example, consider

$$y'(t) + ky(t) = 5e^{-kt}, \qquad y(0) = A.$$

If we let $\hat{y}(s)$ denote the Laplace transform of the solution, $y(t)$, then

$$s\hat{y}(s) - y(0) + k\hat{y}(s) = (s+k)\hat{y}(s) - A$$
$$\text{and} \qquad L[5e^{-kt}] = \frac{5}{s+k}$$
$$\text{so} \quad (s+k)\hat{y}(s) - A = \frac{5}{s+k}$$

and we can solve this equation for $\hat{y}(s)$,

$$\hat{y}(s) = \frac{A}{s+k} + \frac{5}{(s+k)^2}.$$

Now we have to find the function $y(t)$ whose Laplace transform is $\hat{y}(s)$; i.e., we have to find the inverse Laplace transform of $\hat{y}(s)$. In order to do this we observe that the table of transforms we have so far developed allows us to find the inverse Laplace transform of the two parts of $\hat{y}(s)$,

$$L^{-1}\left[\frac{A}{s+k}\right] = AL^{-1}\left[\frac{1}{s+k}\right] = Ae^{-kt}$$

and

$$L^{-1}\left[\frac{5}{(s+k)^2}.\right] = 5L^{-1}\left[\frac{1}{(s+k)^2}.\right]$$
$$= 5te^{-kt}$$

Combining these two results leads to,

$$y(t) = Ae^{-kt} + 5te^{-kt}.$$

An Additional Property of the Transform

A final property of the Laplace transform allows us to find the function whose Laplace transform is the product of two Laplace transforms, $\hat{f}(s)\hat{g}(s)$, where we know the functions $f(t)$ and $g(t)$ whose transforms are $\hat{f}(s)$ and $\hat{g}(s)$. The function whose transform is $\hat{f}(s)\hat{g}(s)$ is not the function $f(t)g(t)$. Instead it is the following,

$$L[(f*g)(t)] = \hat{f}(s)\hat{g}(s)$$
$$\text{where} \qquad (f*g)(t) \;:= \int_0^t f(t-\tau)\,g(\tau)\,d\tau$$

The product, $(f*g)(t)$, is called the **convolution product** of f and g. Life would be simpler if the inverse Laplace transform of $\hat{f}(s)\hat{g}(s)$ was the pointwise product $f(t)g(t)$, but it isn't, it is the convolution product. The convolution product has some of the same properties as the pointwise product, namely

$$(f * g)(t) = (g * f)(t)$$

and $$(h * (f * g))(t) = ((h * f) * g)(t).$$

We will not give the proof of this result but will make use of it nevertheless.

Examples

1. Consider the problem

$$y''(t) + 2y'(t) + 10y(t) = 1, \qquad y(0) = y'(0) = 0.$$

Transforming this problem leads to

$$(s^2 + 2s + 10)\hat{y}(s) = \frac{1}{s},$$

and

$$\begin{aligned}\hat{y}(s) &= \frac{1}{s}\frac{1}{s^2 + 2s + 10}\\ &= \frac{1}{s}\frac{1}{(s+1)^2 + 9}\\ &= \hat{f}(s)\,\hat{g}(s).\end{aligned}$$

We know that $L^{-1}\left[\frac{1}{s}\right] = 1,$ *and* $L^{-1}\left[\frac{3}{(s+1)^2 + 9}\right] = e^{-t}\sin 3t.$ Then by the convolution property of the transform,

$$\begin{aligned}y(t) &= (f * g)(t)\\ &:= \int_0^t f(t-\tau)\,g(\tau)\,d\tau\\ &= \int_0^t 1(t-\tau)\,e^{-\tau}\sin 3\tau\,d\tau\\ &= \int_0^t e^{-\tau}\sin 3\tau\,d\tau\\ &= \frac{3}{10} - \frac{3}{10}e^{-t}\cos(3t) - \frac{1}{10}e^{-t}\sin(3t).\end{aligned}$$

2. Consider now,

$$y''(t) + 2y'(t) + 10y(t) = 1, \qquad y(0) = 6,\ y'(0) = 3.$$

In this case we have

$$s^2\hat{y}(s) - 6s - 3 + 2[s\hat{y}(s) - 6] + 10\hat{y}(s) = \frac{1}{s}$$

or

$$(s^2 + 2s + 10)\hat{y}(s) = \frac{1}{s} + 6s + 15,$$

Then

$$\hat{y}(s) = \frac{1}{s}\frac{1}{(s+1)^2 + 9} + \frac{6s + 15}{(s+1)^2 + 9}$$

Now

$$\frac{6s + 15}{(s+1)^2 + 9} = \frac{6s + 6}{(s+1)^2 + 9} + \frac{9}{(s+1)^2 + 9}$$

The first term on the right above can be inverted using formula 10 from the table of Laplace transforms and the second term can be inverted using formula 9. That is

$$L^{-1}\left[\frac{6s+6}{(s+1)^2+9}\right] = 6L^{-1}\left[\frac{s+1}{(s+1)^2+9}\right] = 6e^{-t}\cos 3t$$
$$L^{-1}\left[\frac{9}{(s+1)^2+9}\right] = 3L^{-1}\left[\frac{3}{(s+1)^2+9}\right] = 3e^{-t}\sin 3t$$

Then the solution to this last initial value problem is

$$\begin{aligned} y(t) &= \frac{3}{10} - \frac{3}{10}e^{-t}\cos(3t) - \frac{1}{10}e^{-t}\sin(3t) + 6e^{-t}\cos 3t + 3e^{-t}\sin 3t \\ &= \frac{3}{10} + 5.7e^{-t}\cos 3t + 2.9e^{-t}\sin 3t \end{aligned}$$

Exercises

Solve each of the following by the Laplace transform:

1. $y''(t) - y'(t) - 2y(t) = 4t^2 + 1$
   $y(0) = 1, \quad y'(0) = -1$

2. $y''(t) - y'(t) - 2y(t) = (t-3)e^{3t}$
   $y(0) = 1, \quad y'(0) = -1$

3. $y''(t) + 25y(t) = 3\cos 2t$
   $y(0) = 1, \quad y'(0) = -1$

4. $y''(t) + 2y'(t) + 26y(t) = 2\sin 5t$
   $y(0) = 2, \quad y'(0) = -2$

5. $y''(t) + y'(t) - 12y(t) = 0$
   $y(0) = 2, \quad y'(0) = 0$

6. $y''(t) + y'(t) - 12y(t) = 0$
   $y(0) = 0, \quad y'(0) = 2$

7. $y''(t) + y'(t) - 12y(t) = 1$
   $y(0) = 0, \quad y'(0) = 0$

8. $y^{(4)}(t) - y(t) = 1,$
   $y(0) = 0, \quad y'(0) = 0 \quad y''(0) = 0, \quad y^{(3)}(0) = 0$

9. $y'(t) + y(t) = e^{t}$
   $y(0) = 2,$

10. $y'(t) + y(t) = e^{-t}$
    $y(0) = 2,$

Use the Laplace transform to evaluate these convolution products:

11. $t * t$

12. $\sin t * t$

13. $1 * 1$

14. $e^{-t} * 1$

Table of Transform Properties

1. $L[C_1 f(t) + C_2 g(t)] = C_1 \hat{f}(s) + C_2 \hat{g}(s)$

2. $L[f(at)] = \frac{1}{a}\hat{f}\left(\frac{s}{a}\right) \quad \textit{for} \quad a > 0$

3. $L[f'(t)] = s\hat{f}(s) - f(0)$

4. $L[f''(t)] = s^2\hat{f}(s) - sf(0) - f'(0) \quad \textit{etc}$

5. $L[tf(t)] = -\hat{f}'(s)$

6. $L[t^2 f(t)] = (-1)^2 \hat{f}''(s) \quad \textit{etc}$

7. $L[H(t-b)f(t-b)] = e^{-bs}\hat{f}(s), \quad \textit{for} \quad b > 0.$

8. $L[e^{bt} f(t)] = \hat{f}(s-b)$

9. $L[(f * g)(t)] = \hat{f}(s)\hat{g}(s) \quad \textit{where} \quad (f * g)(t) := \int_0^t f(t-\tau)g(\tau)\,d\tau$

Table of Laplace Transform Formulas

1. $L[1] = \frac{1}{s} \quad \textit{for } s > 0$

2. $L[e^{bt}] = \dfrac{1}{s-b} \quad \textit{for } s > b.$

3. $L[t^n] = \dfrac{n!}{s^{n+1}}, \; n = 0, 1, 2, \ldots$

4. $L[te^{bt}] = \dfrac{1}{(s-b)^2}.$

5. $L[\sin \omega t].= \dfrac{\omega}{s^2 + \omega^2}.$

6. $L[\cos \omega t].= \dfrac{s}{s^2 + \omega^2}.$

7. $L[t \sin \omega t] = \dfrac{2\omega s}{(s^2 + \omega^2)^2}$

8. $L[t \cos \omega t] = \dfrac{s^2 - \omega^2}{(s^2 + \omega^2)^2}.$

9. $L[e^{bt} \sin \omega t] = \dfrac{\omega}{(s-b)^2 + \omega^2}$

10. $L[e^{bt} \cos \omega t] = \dfrac{s-b}{(s-b)^2 + \omega^2}$

Appendix

Existence and Uniqueness of Solutions

Let

$$L[y(t)] = y''(t) + by'(t) + cy(t)$$

$$S = \{y \in C^2 : L[y] = 0\} = \text{the } \textbf{solution space} \text{ for L}$$

and define

$$W[y_1,y_2](t) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = y_1(t)y_2'(t) - y_2(t)y_1'(t).$$

Here $W[y_1,y_2](t)$ is called the "Wronskian determinant" for the functions y_1 and y_2. Then we can prove the following results:

Lemma 1

If $y_1,y_2 \in S$ then either $W[y_1,y_2](t) = 0$ for all t, or else $W[y_1,y_2](t)$ is never zero for any t.

Proof- Compute the derivative of $W[y_1,y_2](t)$,

$$\tfrac{d}{dt} W[y_1,y_2](t) = \tfrac{d}{dt}(y_1(t)y_2'(t) - y_2(t)y_1'(t))$$

$$= y_1(t)y''_2(t) + y_1'(t)y_2'(t) - y_1'(t)y_2'(t) - y_2(t)y''_1(t)$$

$$= y_1(t)y''_2(t) - y_2(t)y''_1(t) = y_1(t)[-by_2'(t) - cy_2(t)] - y_2(t)[-by_1'(t) - cy_1(t)]$$

$$= -b[y_1(t)y_2'(t) - y_2(t)y_1'(t)] - c[y_1(t)y_2(t) - y_2(t)y_1(t)] = -bW[y_1,y_2](t)$$

where we have used the fact that $y_1,y_2 \in S$. Now

$$\tfrac{d}{dt} W[y_1,y_2](t) = -b\, W[y_1,y_2](t).$$

implies $\quad W[y_1,y_2](t) = C\, e^{bt}$

and $\quad W[y_1,y_2](t) = W[y_1,y_2](t_0)\, e^{-b(t-t_0)}.$

From this last equation we can see that if $W[y_1,y_2](t_0) = 0$, then $W[y_1,y_2](t) = 0$ for every t, and if $W[y_1,y_2](t_0) \neq 0$, then $W[y_1,y_2](t)$ is never zero.■

Lemma 2

Functions $y_1,y_2 \in S$ are independent if and only if $W[y_1,y_2](t) \neq 0$.

Proof- Suppose $y_1, y_2 \in S$ satisfy

$$C_1 y_1(t) + C_2 y_2(t) = 0.$$

Then, by differentiating with respect to t, we see that we also have

$$C_1 y_1'(t) + C_2 y_2'(t) = 0.$$

This set of two equations in two unknowns, $C_1, C_2,$ has a nontrivial solution ($C_1, C_2,$ not both zero) if and only if the determinant of the system is equal to zero. But the determinant of this system is just the Wronskian, $W[y_1, y_2](t),$ which is either zero for all t or zero for no t.
If the Wronskian is zero for all t, then a nontrivial pair of constants exists and in this case $y_1, y_2 \in S$ are dependent.
If the Wronskian is nonzero for all t, then the trivial pair of constants is the only solution and in this case $y_1, y_2 \in S$ are independent.■

Lemma 3
If $y \in S$ then

$$Ce^{-kt} \leq \sqrt{y(t)^2 + y'(t)^2} \leq Ce^{kt}$$

where

$$C = \sqrt{y(0)^2 + y'(0)^2} \qquad and \qquad k = 1 + |b| + |c|.$$

Proof- For $y \in S$ let $U(t) = y(t)^2 + y'(t)^2.$ Then

$$U'(t) = 2y(t)y'(t) + 2y'(t)y''(t) = 2y(t)y'(t) + 2y'(t)[-by'(t) - cy(t)]$$

$$= (2-2c)y(t)y'(t) - 2by'(t)^2,$$

and

$$|U'(t)| \leq 2(1+|c|)|y||y'| + 2|b||y'|^2.$$

Now we use the result that $\quad 2|y||y'| \leq |y|^2 + |y'|^2$

which is a consequence of $\quad (|y| - |y'|)^2 = |y|^2 - 2|y||y'| + |y'|^2 \geq 0.$
Then

$$\begin{aligned}|U'(t)| &\leq (1+|c|)\left(|y|^2 + |y'|^2\right) + 2|b||y'|^2 \\ &\leq 2(1+|b|+|c|)\left(|y|^2 + |y'|^2\right) = 2k\ U(t),\end{aligned}$$

which is the same as saying

$$-2k\ U(t) \leq U'(t) \leq 2k\ U(t).$$

This implies that (check this out),

$$U(0)\,e^{-2kt} \leq U(t) \leq U(0)\,e^{2kt},$$

and

$$\sqrt{U(0)}\, e^{-kt} \le \sqrt{U(t)} \le \sqrt{U(0)}\, e^{kt},$$

which is the estimate we were trying to prove.■

Lemma 4

For arbitrary constants, A, B there exists **at most** one $y \in S$ such that $y(0) = A \text{ and } y'(0) = B.$

Proof- Suppose there are two functions $y_1, y_2 \in S$ which both satisfy $y(0) = A \text{ and } y'(0) = B.$ Then the function $w(t) = y_1(t) - y_2(t)$ must satisfy

$$L[w(t)] = 0 \quad (why?)$$

and $$w(0) = 0 \text{ and } w'(0) = 0, \ (why?).$$

But then Lemma 3 implies $Ce^{-kt} \le \sqrt{w(t)^2 + w'(t)^2} \le Ce^{kt}$ for $C = 0.$ Then $w(t) = w'(t) = 0,$ and $w(t) = y_1(t) - y_2(t) = 0$ and it follows that $y_1(t) = y_2(t).$■

Lemma 5

For arbitrary constants, A, B there exists **at least** one $y \in S$ such that $y(0) = A \text{ and } y'(0) = B.$

Proof- Recall that $L[e^{rt}] = P(r)\,e^{rt} = 0 \ \ implies \ \ P(r) = 0,$ and this polynomial equation has roots $r_1, r_2.$ There are 3 possible cases, in each of which we can find independent solutions y_1, y_2 for the homogeneous equation. That is,

$$if \left\{ \begin{array}{c} r_1 \ne r_2 \text{ both real} \\ r_1 = r_2 \\ r_1, r_2 = a \pm ib \end{array} \right\} \text{ then } \left\{ \begin{array}{c} y_1 = e^{r_1 t}, \quad y_2 = e^{r_2 t} \\ y_1 = e^{r_1 t}, \quad y_2 = t e^{r_1 t} \\ y_1 = e^{at}\cos bt, \ y_2 = e^{at}\sin bt, \end{array} \right\}.$$

In all three cases, we have $$y(0) = C_1 y_1(0) + C_2 y_2(0) = A$$
and $$y'(0) = C_1 y_1'(0) + C_2 y_2'(0) = B$$

and the determinant of this set of two equations in two unknowns is just the Wronskian $W[y_1, y_2].$ Since $y_1, y_2 \in S$ are independent, this determinant is not zero and a unique solution for C_1, C_2 exists for all choices of A,B.■

Combining the last two lemmas allows us to assert that

For arbitrary constants, A, B there exists one and only one solution for

$$L[y(t)] = 0 \quad with \ \ y(0) = A \text{ and } y'(0) = B.$$

Lemma 6

Let $y_1, y_2 \in S$ be linearly independent. Then every $y \in S$ can be written uniquely in the form $y(t) = C_1 y_1(t) + C_2 y_2(t).$

The assertion of this lemma is that there are at least two independent functions in S but any set of three or more functions in S must be dependent. That is, S is a two dimensional subspace of C^2.

Proof- Let $y_1, y_2 \in S$ be linearly independent and, for an arbitrary $y \in S$, let $y(0) = A,\ y'(0) = B.$ Then by lemma 5, there exists at least one choice of C_1, C_2 such that $Y(t) = C_1 y_1(t) + C_2 y_2(t)$ satisfies

$$L[Y(t)] = 0 \quad with \quad Y(0) = A \ and \ \ Y'(0) = B.$$

But by lemma 4, $Y(t) = y(t)$■